A Ring for Description and Control of Time-Delay Systems

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Abstract: - This contribution aims a revision and extension of the ring of retarded quasipolynomial meromorphic functions (R_{MS}) for description and control of time-delay systems (TDS). The original definition has some significant drawbacks – especially, it does not constitute a ring. Our new definition extends the usability to neutral TDS and to those with distributed delays. As first, basic algebraic notions useful for this paper are introduced. A concise overview of algebraic methods for TDS follows. The original and the revised definitions of the ring together with some its properties finish the contribution. There are many illustrative examples that explain introduced terms and findings throughout the paper.

Key-Words: - Time-delay systems, Algebraic description and control, Ring, Coprime factorization, Bézout domain

1 Introduction

Algebraic structures in their charming and attractive elegance proved to be suitable and effective tools for system dynamics description and control system design. Modern control theory has been adopting algebraic approaches and parlance, which are based on TDS description in a suitable field, ring or module and the subsequent operation in the algebraic structure, for decades.

The aim of this paper is to introduce a revise the definition and some basic properties of the R_{MS} ring for description and control of TDS in input-output space, unlike some other methods using state-space domain which prevail. R_{MS} structure was originally introduced in [1]; however, the genesis of the idea can be view already in works of Vidyasagar [2] and Kucera [3] for delayless systems and/or in [4] for TDS. Nevertheless, it has been pointed out in [5] that the structure does not constitute ring. In addition to that, the structure is applicable to retarded systems only and it brings problems when comprising models with distributed delays.

The revised and extended structure can useful when analysis and control of neutral TDS and those with distributed delays. Some stability notions are also discussed and taken into account. Basic properties of the revisited R_{MS} are given for the record as well. To illuminate the ideas and statements, many illustrative examples are introduced throughout the paper.

The paper is organized as follows. Section 2 provides an overview of algebraic notions useful for uninitiated readers to comprehend the rest of the contribution. A non-exhaustive introduction to algebraic structures and methods used in description, analysis and control of TDS can be found in Section 3. The original and a revised definition of R_{MS} are the contents of Section 4. Section 5 includes a list of selected properties of the now conception supported by some examples. Section 6 concludes the paper and outlines the usability of the R_{MS} ring.

2 Basic Algebraic Notions

Prior to a brief overview of particular algebraic structures utilized by some authors when analysis (and/or synthesis) of TDS, it is convenient to introduce some basic algebraic notions being used in this paper and their elementary properties if useful, see e.g. [6], [7].

A *group*, G, is an algebraic structure with binary operation $\cdot$ satisfying:

a) For each $a,b \in G$, it holds that $a \cdot b \in G$.

b) For all $a,b,c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c) \in G$ (associativity).

c) There exists an element $e \in G$, such that for every element $a \in G$, it holds that $a = a \cdot e = e \cdot a \in G$ (*identity element, neutral element*).

d) For each $a \in G$, there exists an element $b \in G$ such that $a \cdot b = b \cdot a = e \in G$ (*inverse element*).

A set satisfying a) and b) only from the definition above, i.e. without a necessity of identity and inverse elements, is called a *semigroup*. If one requires the existence of an identity element, so-called *monoid* is obtained. A group with the commutative property, i.e.

e) For each $a, b \in G$, $a \cdot b = b \cdot a \in G$

is called a commutative (abelian) group.

A *ring*, *R*, is a set with two binary operations +, · (generally interpreted summation and addition) for which it holds true the following:

a) *R* is a commutative group under addition with an identity element denotes as 0.

b) For any $a, b, c \in R$, $(a+b)\cdot c = a \cdot b + b \cdot c \in R$ and $c \cdot (a+b) = c \cdot a + c \cdot b \in R$ (left and right distributivity).

c) For every $a, b, c \in R$, it holds that $(a \cdot b)\cdot c = a \cdot (b \cdot c) \in R$ (Associativity of multiplication).

Some authors add another property of a ring as:

d) There exists $1 \in R$ such that for every $a \neq 0 \in R$, $a \cdot 1 = 1 \cdot a \in R$ (multiplicative identity).

If d) holds, then a ring is a commutative group under + and a commutative monoid under ·, together with distributivity. In a *commutative ring*, the commutative property holds also for multiplication.

A *unit* of the ring (or an *invertible element*) is $a \neq 0 \in R$, for which there exists $a^{-1} \in R$, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. If all elements of a ring are units, the ring is called a *field*.

It is said that $b \in R$ *divides* $a \in R$ (i.e. $b \mid a$) if there exists $q \in R$, such that $a = q \cdot b$. Two elements $a, b \in R$ are *associated* if $b \mid a$ and $a \mid b$.

Let *R* be a commutative ring and $a, b \in R$. A *common divisor* $c \in R$ of *a*, *b* is an element of the ring, for which $c \mid a$ and $c \mid b$. $d \in R$ is the *greatest common divisor* (GCD) of *a*, *b* if for every common divisor $c \in R$ of $a, b \in R$ it holds that $c \mid d$. The CGD is determined unambiguously except for associativity.

A nonzero noninvertible element a of a commutative ring R is called *irreducible* if it is divisible solely by a unit or any element associated with a. In some rings, so-called *prime elements* generalizing prime numbers are introduced. A prime elements is a nonzero noninvertible $a \in R$, such that if $a \mid (b \cdot c)$ for some $b, c \in R$, then always $a \mid b$ or $a \mid c$. Every prime element is irreducible, the converse is not true in general.

A ring *R* in which every nonzero noninvertible $a \in R$ can be uniquely decomposed in a (finite) product of irreducible or prime elements (except for the ordering and associativity) is called a unique factorization ring (UFR).

A commutative ring with identity (under multiplication) such that for any two elements $a \neq 0 \in R$ and $b \neq 0 \in R$ it holds that $a \cdot b \neq 0$ is called an *integral domain*. An URF which is an integral domain is labeled as a unique factorization domain (UFD).

A *field of fractions* of an integral domain *R* (at least with one element) is the "smallest" field containing *R*, such that necessary elements satisfying the divisibility (by a nonzero element) are added. An element c of this field can be expresses in the form $c = a / b$ where $a, b \in R$, $b \neq 0$.

An *ideal I* (of the ring *R*) is a subset of *R* with the following properties:

a) For every $a, b \in I$, it holds that $a + b \in I$.

b) For each $a \in I$ and $r \in R$, $a \cdot r \in I$.

It holds that an intersection of ideals is an ideal as well. Let $M = \{a_1, a_2, \dots a_n\} \subseteq R$, then an intersection of all ideals of *R* containing *M* is called an ideal *generated by M*. It is also the "smallest" ideal including *M*. Ideals of the form $aR = \{a \cdot r \mid r \in R\}$, i.e. those generated by (the only one) element a are called *principal*.

If every ideal of an integral domain is principal, so-called *principal ideal domain* (PID) is obtained. It holds true that every PID is UFD; however, the converse is not true in general.

A *Noetherian* ring *R* is primarily defined as that satisfying the so-called finite ascending chain condition. Equivalently, it is possible to circumscribe the term as follows: A ring *R* is Noetherian if its every ideal is finitely generated, i.e. $n = |M|$ is a finite number.

A (left) *module* (or *R-module*) *M* over the ring *R* is a commutative group satisfying:

a) For every $r \in R$, $a, b \in M$, it holds that $r \cdot (a+b) = r \cdot a + r \cdot b \in M$.

b) For every $r, s \in R$, $a \in M$, $(r+s)\cdot a = r \cdot a + s \cdot a \in M$.

c) For every $r, s \in R$, $a \in M$, $(r \cdot s)\cdot a = r \cdot (s \cdot a) \in M$.

d) If there exists a multiplicative identity $1 \in R$, and $a \in M$, then $1 \cdot a = a \in M$

Modules are similar to vector spaces, yet in modules, coefficients are taken from rings, not from fields. A *free module* is that with a basis. For instance, since nonzero elements in a ring are not necessarily invertible, a relation $\sum_{i=1}^{n} r_i \cdot a_i = 0, r_i \in R, a_i \in M$, where M is a free module, does not imply that each r_i is the linear combination of the remaining ones (Conte and Perdon, 2000).

A *partially ordered set* (poset) is defined as an ordered pair $P = (S, \preceq)$ where S is called the ground set of P and $\preceq$ is the partial order of P. A relation $\preceq$ is a poset on S if:

a) For all $a \in S$, $a \preceq a$ (reflexivity)

b) For $a, b \in S$, if $a \preceq b$ and $b \preceq a$, then $a \equiv b$ (antisymmetry)

c) For $a, b, c \in S$, $a \preceq b$ and $b \preceq c$ implies $a \preceq b$ (transitivity)

From a PID, a *Bézout domain* is distinguished in which every *finitely* generated ideal is principal. In a Bézout domain, PID is UFD and viceversa. Thus, a PID admits the existence of an infinitely generated ideal which is principal.

In a Bézout domain *R*, for every pair $a, b \in R$ (or generally for a finite set of elements) there exists the GCD which meets the *Bézout identity* (or more generally a *linear Diophantine equation*)

$$a \cdot x + b \cdot y = \mathrm{GCD}(a, b),\ x, y \in R \tag{1}$$

A solution $x, y \in R$ is not determined uniquely but (an infinitely many) solutions of (1) are given by the parameterization

$$x = x_0 \pm z \cdot \frac{b}{\mathrm{GCD}(a,b)},\ y = y_0 \mp z \cdot \frac{a}{\mathrm{GCD}(a,b)} \tag{2}$$

where $\{x_0,\ y_0\}$ is a particular solution of (1) and $z \in R$

If (1) is solved for any $c \in R$ on the right-hand side instead of $\mathrm{GCD}(a,b)$, it is necessary to verify whether there exists $\mathrm{GCD}(a,b)$ (especially in a ring which is not Bézout or PID) for which $\mathrm{GCD}(a,b) \mid c$.

The Bézout identity can be solved e.g. using a *generalized* (extended) *Euclidean algorithm* which can be described as follows. Let a, b be given and the task is to find $d = \mathrm{GCD}(a,b)$ and a pair x, y according to (1). The iterative procedure can be written as follows

$$\begin{aligned} & r_i = r_{i-2} - \lfloor q_i \rfloor \cdot r_{i-1},\ r_{i-2} \geq r_{i-1} \geq r_i \\ & i = 3, 4, \ldots, n \end{aligned} \tag{3}$$

i.e. the current reminder r_i of the division can be expressed by preceding reminders r_{i-1}, r_{i-2} and using the whole quotient q_i.

In every step of the algorithm, it is possible to write the following identity

$$r_i = a \cdot x_i + b \cdot y_i \tag{4}$$

where x_i, y_i are from the ring. The first two reminders are chosen as

$$\begin{aligned} & r_1 = a = a \cdot 1 + b \cdot 0 \\ & r_2 = b = a \cdot 0 + b \cdot 1 \end{aligned} \tag{5}$$

The desired $d = \mathrm{GCD}(a,b)$ then equals the last nonzero reminder, $r_n \neq 0,\ n < \infty$.

The whole procedure can be expressed in a table (matrix) form as follows

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right] \sim \begin{array}{c} \text{elementary} \\ \text{matrix} \\ \text{operations} \end{array} \sim \left[\begin{array}{cc|c} v & t & 0 \\ x & y & d \end{array}\right] \tag{6}$$

The result is determined by two Diophantine equations

$$\begin{aligned} & a \cdot v + b \cdot t = 0 \\ & a \cdot x + b \cdot y = d \end{aligned} \tag{7}$$

In the case when (1) is solved for any fixed $c \in R$ on the right-hand side instead of $d = \mathrm{GCD}(a,b)$ it is possible (if a solution exists) to use the extended Euclidean algorithm again in the following two possibilities:

1) To use scheme (6) for $c \in R$ instead of $d = \mathrm{GCD}(a,b)$. Generally, it is not necessary to achieve the zero element on the upper right matrix corner.

2) Obviously

$$\begin{aligned} & a \cdot x + b \cdot y = \mathrm{GCD}(a,b) \quad / \frac{c}{\mathrm{GCD}(a,b)} \\ & a \frac{xc}{\mathrm{GCD}(a,b)} + b \frac{yc}{\mathrm{GCD}(a,b)} = c \\ & a x_1 + b \cdot y_1 = c \end{aligned} \tag{8}$$

Hence, $\mathrm{GCD}(a,b)$, x, y are found using (6) first, and subsequently, the following substitution is used

$$x_1 = x\frac{c}{\mathrm{GCD}(a,b)},\ y_1 = y\frac{c}{\mathrm{GCD}(a,b)} \tag{9}$$

to get the desired solution.

For the necessity and comprehension of the further text, some basic notions from the complex functions analysis ought to be introduced as well.

A *holomorphic function* is a complex-valued function of a single (or multiple) complex variable defined on a region $\mathbb{D} \subseteq \mathbb{C}$ which is infinitely complex differentiable (i.e. there exists all complex derivatives) at any point $z_0 \in \mathbb{D}$.

The term holomorphic function is often used interchangeably with or compared to an *analytic function* which is generally a complex-valued function of a single (or multiple) complex variable defined on a region $\mathbb{D} \subseteq \mathbb{C}$, in which the Taylor series expansion exists at every point $z_0 \in \mathbb{D}$. That is, a series $T(z) = \frac{1}{i!}\sum_{i=0}^{\infty} f^{(i)}(z_0)(z - z_0)^i$ converges to $f(z)$ for every point z from a neighborhood of z_0. For complex functions, a holomorphic function implies an analytic function. A function holomorphic on all $\mathbb{C}$ is called *entire*.

An *isolated singularity* of a complex function $f(z)$ is a point z_0, in which the function is not differentiable; however, there exists an open disk D centered at z_0 such that $f(z)$ is holomorphic on the disk excluding z_0. There are several types of isolated singularities. A *pole* is an isolated singularity z_0 of $f(z)$ such that $f(z)$ converges uniformly to infinity for $z \to z_0$. Thus, if there exists the improper limit $\lim_{z \to z_0} f(z) = \infty$, then there exists also $n \in \mathbb{N}$, so that $\lim_{z \to z_0} (z - z_0)^n f(z) < \infty$. A *removable singularity* is another type of an isolated one for which $\lim_{z \to z_0} f(z) \neq \infty$. In this case, it is possible to define $f(z_0) = \lim_{z \to z_0} f(z)$, so that $f(z)$ becomes holomorphic. An *essential singularity* represents the last type of an isolated singularity which evinces "peculiar" behavior within the neighborhood of the singularity, and it holds that the limit $\lim_{z \to z_0} f(z)$ does not exist here.

A *meromorphic function* is a complex-valued function of a complex variable which is holomorphic on an open subset $\mathbb{D} \subseteq \mathbb{C}$ except a set of poles. The function can be expressed as a ratio of two holomorphic functions.

3 Fields, Rings and Modules for Description and Control of TDS

The nascence of algebraic methods in description of TDS is connected with fields, namely with systems over fields [9], which can be written in the (retarded) state-space form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{10}$$

where elements of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are from a fixed field and $\dot{\mathbf{x}}(t) = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}$.

The next step was to further generalize the concept of linear systems, to include the case in which coefficients belong to a ring. The first, general, in-depth research into the properties of systems over rings was constituted in [10], [11]. One of the primordial attempts to utilize ring theory to infinite-dimensional linear systems was made by Kamen [12] where an operator theory was presented, the particular case of systems defined via rings of distributions. Namely, the ring Θ generated by the entire functions $\theta_\sigma(s)$ defined as

$$\begin{aligned} &\varphi_\sigma(s) = 0.5(\theta_\sigma + \theta_{\bar{\sigma}}), \psi_\sigma(s) = 0.5\mathrm{j}(\theta_\sigma - \theta_{\bar{\sigma}}) \\ &\theta_\sigma(s) = \frac{1 - \exp(-\tau(s - \sigma))}{s - \sigma}, \sigma \in \mathbb{C} \end{aligned} \tag{11}$$

and their derivatives and 1 was introduced there. Ring models for TDS with lumped delays was published in [13].

In [14], linear systems over commutative rings, especially TDS, were intensively studied. The author i.a. presented the simplest TDS over rings, those with commensurate delays where the introduction of the operator $\delta x(t) := x(t - \tau)$, where τ represents the smallest delay, yields state matrix entries in the ring of polynomials R[δ]. In more details, let the model be

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{k=0}^{N} \mathbf{A}_k \mathbf{x}(t - k\tau) + \mathbf{B}_k \mathbf{u}(t - k\tau) \\ \mathbf{y}(t) &= \sum_{k=0}^{N} \mathbf{C}_k \mathbf{x}(t - k\tau) \end{aligned} \tag{12}$$

then state and output matrices in (10) read

$$\mathbf{A} = \sum_{k=0}^{N} \mathbf{A}_k \delta^k, \mathbf{B} = \sum_{k=0}^{N} \mathbf{B}_k \delta^k, \mathbf{C} = \sum_{k=0}^{N} \mathbf{C}_k \delta^k \tag{13}$$

Using a substitution $\delta^k \to \exp(-k\tau s)$, one can obtain the Laplace transform form of the state model for TDS with commensurate delays. If delays are not commensurate, we need to define a finite set of delay operators $\delta_1, \delta_2, ..., \delta_N$ resulting in a ring $\mathrm{R}[\delta_1, \delta_2, ..., \delta_N]$. Some authors, e.g. Youla [15], introduced the field $\mathrm{R}(\delta_1, \delta_2, ..., \delta_N)$ of rational functions in $\mathrm{R}[\delta_1, \delta_2, ..., \delta_N]$ in order to study networks with transmission lines (i.e. delayed system). Reachability and observability of a general system with coefficients over a ring are analyzed in [14] as well.

Conte and Perdon in [16] further studied the realization of such systems. These authors also developed the geometrical approach to the study of dynamical systems with coefficients over a ring concerning TDS. The overview of the methodology was presented in [8]. In this framework, the main tool lies in the view that $\mathbf{x}(t), \mathbf{u}(t), \mathbf{y}(t)$ in (10) are free *R*-modules.

Concerning input-output maps, which are substantive for the aim of this paper, the conception of 2-D systems which naturally arises from the transfer function of a TDS with commensurate delays over a ring (12), (13) was introduced in [13], [14]. Translating the state-space description into the transfer function results in a rational function in s and $\exp(-\tau s)$. This expresses that two operators are used here, i.e. the integrator and the delay operator, which are algebraically independent (due to the fact that the exponential term is a transcendental function) in the meaning of that there is no nontrivial linear combination of s and $\exp(-\tau s)$ over real numbers equals to zero. Thus, the ring $\mathrm{R}[s, \exp(-\tau s)]$ of quasipolynomials, which is isomorphic to the ring of real polynomials in two variables (a so-called 2-D polynomial) $\mathrm{R}[s, z]$, is obtained. Quasipolynomials defined here are connected with commensurate delays. This concept was further studied and developed e.g. in [17], [18].

However, some authors pointed out that the use quasipolynomials does not permit to effectively handle some stabilization and control tasks, thus other rings based on quasipolynomials for TDS with commensurate delays were introduced.

For instance, in [4], [19] there were established the following rings: A ring $\mathcal{R} = \Theta \cup \mathrm{R}[\exp(-\tau s)] = \Theta[\exp(-\tau s)]$ of all linear combinations, with real coefficients, of distributed delays from Θ and lumped delays, and a ring $\mathcal{E} = \mathrm{P}[s] = \Theta \cup \mathrm{R}[s, \exp(-\tau s)]$ of so-called *pseudopolynomials* which consists of Laplace transforms of operators that are generated using derivatives, lumped and distributed delays. Any element $T(s) \in \mathcal{E}$ can be written in the (coprime) form $T(s) \in N(s, \exp(-\tau s))/D(s)$, $N(s, \exp(-\tau s)) \in \mathrm{R}[s, \exp(-\tau s)]$, $D(s) \in \mathrm{R}[s]$. Two pseudopolynomials are coprime if and only if there are neither their common zeros nor factors in the form $\exp(-k\tau s)$. Ring $\mathcal{R}[s]$ is not isomorphic to $\mathcal{R}[x]$, which means that the variables are not algebraically independent (transcendental) over $\mathcal{R}$, see an example in [4]. Moreover, it is a Bézout domain, yet not a Euclidean ring nor a Noetherian ring nor a UFD. Notice that $\mathcal{E}$ and $\mathrm{R}[s, \exp(-\tau s)]$ share the same field of fractions, i.e. $\mathrm{R}(s, \exp(-\tau s))$. The transfer function can then be expresses as a fraction of two pseudopolynomials.

Behavioral approach, as it was introduced for dynamical systems in [20], was presented by [21] for TDS, again with commensurate delays. In contrast to above mentioned works, the author considered systems in the behavioral point of view instead of systems over rings. A behavior is the kernel of a delay-differential operator. More precisely, consider equations in the scalar case in the form

$$\sum_{j=0}^{L} \sum_{i=0}^{N} p_{ij} x^{(i)}(t - j) = 0 \tag{14}$$

where $p_{ij}, t \in \mathrm{R}$, $x^{(i)}(t)$ denotes the *i*-th derivative of the function $x(t)$: $\mathrm{R} \to \mathrm{R}$. Behaviors $\mathcal{B}$ are those functions $x(t)$ satisfying (14). Alternatively, $\mathcal{B} = \ker \widetilde{P}$ where $P = \sum_{j=0}^{L} \sum_{i=0}^{N} s^i z^j \in \mathrm{R}[s, z]$ and $\widetilde{P}$ denotes the associated delay-differential operator, i.e. $\widetilde{P} x(t) = \sum_{j=0}^{L} \sum_{i=0}^{N} p_{ij} x^{(i)}(t - j)$. It is stated in [21] that it is algebraically more adequate to consider the ring $\mathrm{R}[s, z, z^{-1}]$ instead of $\mathrm{R}[s, z]$. There is also defined the ring

$$\mathcal{H} := \{ p \in \mathrm{R}(s)[z, z^{-1}] \mid p(s, z) \in H_C \tag{15}$$

as the appropriate domain in order to translate relations between behaviors, lying between $\mathrm{R}[s, z, z^{-1}]$ and $\mathrm{R}(s)[z, z^{-1}]$, where the latter means the ring of polynomials in z, z^{-1} with the coefficients in rational functions in s with real parameters, and H_C is the set of all entire functions.

It was proved that $\mathcal{H}$ is not UFD and not a Noetherian ring; however, it is a Bézout ring.

However, delays are naturally real-valued and thus the limitation to commensurate delays is rather restrictive for real applications [22]. Dealing with rings for input-output maps of TDS with even non-commensurate delays, it is crucial for this paper to mention here the family of approaches (originally developed for delayless systems) utilizing a field of fractions where the transfer function is expressed as a ratio of two coprime (or relatively prime) elements of a suitable ring [2], [3], [23]. The process of finding such coprime pair is called a *coprime factorization*.

One of such rings for continuous-time systems is a ring of stable and proper rational functions, R_{PS} [3], [24]. An element of this ring is defined as a ratio of two polynomials in s over $\mathbb{R}$ where the denominator polynomial is Hurwitz stable (i.e. free of roots located in the closed right-half plane including imaginary axis) and, moreover, the ratio is proper (i.e. the s-degree of the numerator is less or equal to the denominator). Alternatively, the element of R_{PS} is analytic and bounded for $\operatorname{Re} s \geq 0$ including infinity, i.e. it lies in $H_\infty(\mathbb{C}^+)$. Such a definition is, however, not sufficient for TDS since e.g. the Laplace form of a stable system including in $H_\infty(\mathbb{C}^+)$ can have an unstable denominator.

The utilization of R_{PS} in description (and control) of TDS requires a rational approximation of a general meromorphic transfer function as a first step of a coprime factorization, for instance, by a substitution of the exponential terms, $\exp(-\tau s) \approx X(s) \in \mathbb{R}(s)$, see e.g. [25], [26]. A similar idea, yet over $\mathbb{R}[s]$ was presented e.g. in [27].

An example of a coprime factorization in R_{PS} follows.

Example 1. Consider a stable TDS with distributed delays governed by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1 - \exp(1)\exp(-s)}{s - 1} \tag{16}$$

Use of, e.g., the first order Padé rational approximation results in

$$G(s) = \frac{Y(s)}{U(s)} \approx \frac{0.5s(1 + \exp(1)) + 1 - \exp(1)}{(s - 1)(0.5s + 1)} = \frac{b(s)}{a(s)} \tag{17}$$

where $a(s)$, $b(s) \in \mathbb{R}[s]$. Notice that the common root $s = -1$ (removable singularity) characterizing the delay distribution in this example vanished after the rationalization. An addition, although the relative order of the transfer function is preserved, the absolute one has increased. To establish coprime factors $A(s) = a(s)/m(s)$, $B(s) = b(s)/m(s)$, $m(s) \in \mathbb{R}[s]$ (with no zero in $\mathbb{C}^+$), $A(s) \in R_{PS}$, $B(s) \in R_{PS}$, one has to realize the divisibility condition in R_{PS}: Any $A(s) \in R_{PS}$ divides $B(s) \in R_{PS}$ if and only if all unstable zeros (including $s \to \infty$) of $A(s)$ are those of $B(s)$. Inclusion of infinity in the condition gives rise to the requirement $\deg m(s) = \deg a(s) = 2$, and moreover, there is no s with $\operatorname{Re} s \geq 0$ satisfying $m(s) = 0$. ■

The main drawback of the ring, i.e. the necessity of a rational approximation, induces the idea of introduction a similar, yet rather different, ring avoiding this operation.

4 R_{MS} Ring

4.1 Original definition

The original definition of the ring of proper and stable retarded quasipolynomial (RQ) meromorphic functions, R_{MS}, is the subject of this subsection [1]. The basic idea for its introduction proceeds from the following ideas. First, as mentioned above in the previous section, a rational approximation of the transfer function in the form of a ratio of two quasipolynomials is required for the use of the ring R_{PS}. This operation brings a loss of system dynamics information, as can be seen from Example 1. Second, from the practical point of view, there is no reason to be limited to commensurate delays in a model, thus, a more universal description ought to be introduced. Third, authors took into account the fact that two variables, z and s, are not independent from the functional point of view, thus, a one-dimensional (1-D) instead of 2-D approach can be used. Last but not least, as stated above, quasipolynomials in the transfer function do not permit to effectively handle some stabilization and control tasks such as impulse-free stability and controller properness and parameterization.

Definition 1 (R_{MS} ring – original). An element $T(s) \in R_{MS}$ is represented by a proper fraction of two quasipolynomials

$$T(s) = \frac{y(s)}{x(s)} \tag{18}$$

where a denominator $x(s)$ is a quasipolynomial of degree n and a numerator can be factorized as

$$y(s) = \tilde{y}(s)\exp(-\tau s) \quad (19)$$

where $\tilde{y}(s)$ is a quasipolynomial of degree l and $\tau \geq 0$. $x(s)$ is stable, which means that there is no zero of $x(s)$, s_0, such that $\operatorname{Re} s_0 \geq 0$. Moreover, the ratio is proper, i.e. $l \leq n$. ■

Obviously, the condition $\tau > 0$ is too restrictive (or more likely a misprint); the inequality $\tau \geq 0$ would be more natural instead. The original definition of R_{MS} has some drawbacks; especially, it does not constitute a ring, which requires making some changes in the definition. Namely, although the retarded structure of TDS is considered only, the minimal ring conditions require the use of neutral quasipolynomials at least in the numerator of $T(s)$. Moreover, the original definition brings problems when comprising models with distributed delays and handling a coprime factorization.

4.2 H_∞ and BIBO stability

To comprehend the revisited definition, notion of H_∞, BIBO, formal and strong stability have to be briefly introduced first.

A system is H_∞ stable if its transfer function $G(s)$ lies in the space $H_\infty(\mathbb{C}^+)$ of functions analytic and bounded in the right-half complex plane, i.e. providing the finite norm

$$\|G\|_\infty := \sup\{G(s) : \operatorname{Re} s \geq 0\} < \infty \quad (20)$$

see e.g. [27]. That is, the system has finite $L_2(0,\infty)$ to $L_2(0,\infty)$ gain where $L_2(0,\infty)$ norm of an input or output signal $h(t)$ is defined as

$$\|h(t)\|_2 := \sqrt{\int_0^\infty |h(t)|^2 \,\mathrm{d}t} \quad (21)$$

Notice, for instance, that a transfer function having no pole in the right-half complex plane but a sequence of poles with real part converging to zero can be H_∞ unstable due to unbounded gain at the imaginary axis [28].

The notion of *BIBO* (Bounded Input Bounded Output) stability is stronger than the preceding one and usually more difficult to analyze. A single-input single-output (SISO) TDS is BIBO stable if a bounded input $|u(t)| < M_1$, $t < 0$, $M_1 \in \mathbb{R}$ produces a bounded output $|y(t)| < M_2$, $t < 0$, $M_2 \in \mathbb{R}$; in other words, it has a finite L_∞ gain. It holds that the system is BIBO stable if its transfer function is an element of a commutative Banach algebra $\Lambda(L_1 + \mathbb{R}\delta)$ of Laplace transforms of functions of the form

$$h(t) = h_a(t) + \sum_{i=1}^{\infty} h_i \delta(t - \tau_i), t \geq 0 \quad (22)$$

where $h_a(t) \in L_1(0,\infty)$, i.e.

$$\int_0^\infty |h_a(t)| \mathrm{d}t < \infty \quad (23)$$

$h_i \in \mathbb{R}$, $\tau_0 = 0, \tau_i > 0,$ for $i > 0$, $\delta(t)$ stands for the Dirac delta function, and

$$\sum_{i=1}^{\infty} |h_i| < \infty \quad (24)$$

BIBO stability implies H_∞ stability [29], [30].

Formal stability of neutral TDS is defined in the state-space domain and this theory is going beyond the topic of this paper. However, it can be formulated simply as follows: formal stability means that the system has only a finite number of poles in the right-half complex plane [31]. In other words, the rightmost vertical strip of poles does not reach or cross the imaginary axis.

The feature of a neutral TDS that the position of the rightmost vertical strip is not continuous in real axis is not continuous [32] gives rise to another (yet a germane) stability notion. *Strong* stability means that the strip remains in $\mathbb{C}_0^-$ when subjected to small variations in delays (i.e. a TDS remains formally stable). Although this stability notion is defined in state-space domain, the following input-output test can be performed

$$\sum_{j=1}^{h_n} |m_{nj}| < 1 \quad (25)$$

see e.g. [33], [34] where m_{nj} are coefficients for the highest s-power in the characteristic quasipolynomial (transfer function denominator)

$$m_0(s) = s^n + \sum_{i=0}^{n} \sum_{j=1}^{h_i} m_{ij} s^i \exp(-s\eta_{ij}), \eta_{ij} \geq 0 \quad (26)$$

4.3 Revised definition

The following simple example shows that the original definition does not constitute a ring.

Example 2. Consider two elements of R_{MS}

$$T_1(s)=\frac{s}{s+2}, T_2(s)=\frac{(s+1)\exp(-s)}{s+2} \tag{27}$$

Yet, a sum of them

$$\begin{aligned}T(s)&=T_1(s)+T_2(s)\\&=\frac{s(1+\exp(-s))+\exp(-s)}{s+2}\notin R_{MS}\end{aligned} \tag{28}$$

since the numerator is a neutral (even formally unstable) quasipolynomial, which is inconsistent with the original ring definition. ■

The above introduced example indicates that it is necessary to include neutral terms in the definition.

The second drawback comes from the requirement of stable denominator. The transfer function of a stable TDS with distributed delays has common numerator and denominator root from the right-half plane; however, there is no reason to consider it as unstable in any sense, see e.g. stable system (16). Rephrased, an element of the ring should include a removable singularity in $\mathbb{C}^+$ (but not poles). Analogously, spectral stabilizability can be viewed in the similar manner [35].

Because of this, $H_\infty(\mathbb{C}^+)$ seems to be a suitable candidate for the ring definition (as for R_{PS} ring).

However, there are some troubles with neutral systems, namely, although a formally unstable neutral TDS with a vertical strip of poles tending to the imaginary axis from left (for $\operatorname{Im} s_0 \to \infty$) can be BIBO (and hence $H_\infty(\mathbb{C}^+)$) stable, it does not permit the so called Bézout factorization, [28], [30]. Any two elements $A(s), B(s) \in H_\infty(\mathbb{C}^+)$ form a Bézout (coprime) factorization if and only if

$$\inf_{\operatorname{Re} s\geq 0}(|A(s)|+|B(s)|)>0 \tag{29}$$

i.e. there exist (a stabilizing coprime pair) $Q(s), P(s) \in H_\infty(\mathbb{C}^+)$, such that

$$A(s)P(s)+B(s)Q(s)=1 \tag{30}$$

Example 3. A TDS of neutral type has a transfer function

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b(s)}{a(s)}=\frac{1}{(1-\exp(-s))(s+1)} \tag{31}$$

Clearly, a pair

$$B(s)=\frac{1}{s+2}, A(s)=\frac{(1-\exp(-s))(s+1)}{s+2} \tag{32}$$

has no nontrivial (non-unit) common factor, i.e. it is coprime. However, $|A(\pm k2\pi j)|=0, k\in\mathbb{N}$, and $\lim_{k\to\infty}|B(\pm k2\pi j)|=0$, hence (29) does not holds true and the system is not Bézout coprime nor BIBO stabilizable. ■

As stated in [35] for neutral-type TDS, a system that is not formally stable is not BIBO stable nor stabilizable. However, this is not true exactly, as shown in [28].

Since formal stability is not given in input-output relation (transfer function), consider a rather more strict notion – strong stability – given by condition (25) instead. Formal stability is hence required; however, its testing by strong stability condition (25) could not be included in the ring definition since it may lead to strong instability when algebraic operations on ring elements.

The following short examples demonstrate and clarify the above ideas.

Example 4. Let be given three neutral delayed systems (plants) governed by transfer functions

$$\begin{aligned}G_1(s)&=\frac{1}{s+s\exp(-s)+1},\\G_2(s)&=\frac{1}{(s+s\exp(-s)+1)(s+1)},\\G_3(s)&=\frac{1}{(s+s\exp(-s)+1)(s+1)^4}\end{aligned} \tag{33}$$

All the systems have poles located in the "stable" half-plane $\mathbb{C}_0^-$, except for $\operatorname{Im} s_0 \to \infty$ where the asymptotic pole lies on the imaginary axis, see Fig. 1, where displayed poles (blue asterisks) are -0.4011, -0.0379 + 3.4264j, -0.0054 + 9.5293j, -0.0020 + 15.7713j, -0.0010 + 22.0365j, -0.0006 + 28.3096j, -0.0004 + 34.5864j, -0.0003 + 40.8652j, -0.0002 + 47.1451j. (2.67)

However, although there is no pole (except the asymptotic case) in $\mathbb{C}^+$, neutral systems (33) can not be considered as asymptotically stable since the is no positive α satisfying $\operatorname{Re} s_0 \leq -\alpha$ for all s_0, which is necessary for stability of neutral TDS.

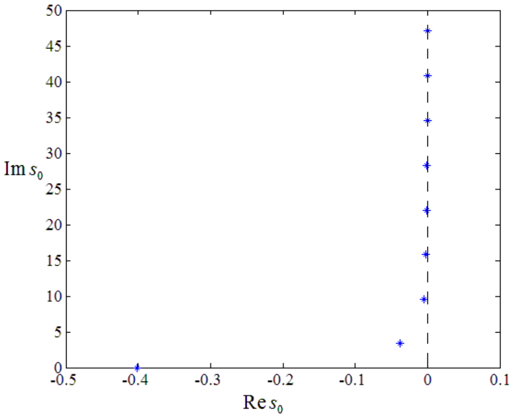

Fig. 1. Root loci of the rightmost poles of $G_1(s)$ from (33)

Moreover, these systems are neither strongly nor formally stable, simply, the chain of poles reaches the imaginary axis. Nevertheless, other stability notions are more attractive. An easy test on $G_1(\mathrm{j}\omega)$, $G_2(\mathrm{j}\omega)$, $G_3(\mathrm{j}\omega)$ shows that $\|G_1\|_\infty = \infty$, $\|G_2\|_\infty = 2$, $\|G_3\|_\infty = 1$, hence $G_1 \notin H_\infty(\mathrm{C}^+)$, $G_2, G_3 \in H_\infty(\mathrm{C}^+)$. As proved in [27], G_1 and G_2 are not BIBO stable, yet G_3 is BIBO stable. This means that formal instability does not automatically implies H_∞ or BIBO instability which makes problems when decision about the inclusion of the system into an algebraic structure (or set). ■

Example 5. This example demonstrates the necessity of formal stability in the definition of R_{MS} ring, not only for elements of R_{MS} but also for their inversions.

Consider a coprime factorization in $H_\infty(\mathrm{C}^+)$ of system $G_2(s)$ from (33), i.e.

$$G(s) = \frac{\dfrac{1}{(s+2)^2}}{\dfrac{[(1+\exp(-s)s+1)(s+1)]}{(s+2)^2}} = \frac{B(s)}{A(s)} \tag{34}$$

More information about (Bézout) coprime factorization can be found in Section 5. Notice that the factorization (34) is coprime yet not Bézout.

As stated above, the system $G(s)$ is formally unstable but from $H_\infty(\mathrm{C}^+)$, i.e. $B(s)/A(s) \in H_\infty(\mathrm{C}^+)$. However, one can verify that $1/A(s) \notin H_\infty(\mathrm{C}^+)$. This yields a mismatch in the ring definition since there is not an unambiguous answer whether $A(s)$ is invertible (a unit) or not. If both terms were not coprime, it would not pose a problem since such situations are natural also in R_{PS} ring. If $G(s)$ was formally stable, it would hold that $1/A(s) \in H_\infty(\mathrm{C}^+)$. As a conclusion, a set $H_\infty(\mathrm{C}^+)$ is not a sufficient candidate for R_{MS} ring. ■

Hence, there seem to be two possibilities for the ring definitions regarding formal stability. Either to include the requirement of formal stability of the quasipolynomial numerator in the ring definition and thus to exclude the existence of (Bézout) coprime factorization for formally unstable systems, or to take it into consideration in ring divisibility conditions. Naturally, we decided to choose the latter option, since it is not possible to avoid a formal unstable numerator in ring elements as demonstrated in Example 2.

Example 6. The aim of this example is to show that strong stability could not be included in the ring definition; however, the necessity of formal stability has been already proved in Example 5.

Consider a formally and strongly stable element from $H_\infty(\mathrm{C}^+)$

$$T(s) = \frac{1}{(1+\exp(-0.8s))s+1} \tag{35}$$

Now make a multiplication

$$T_2(s) = T(s)T(s) = \frac{1}{[(1+\exp(-0.8s))s+1]^2}$$
$$= \frac{1}{(1+\exp(-1.6s)+2\exp(-0.8s))s^2+2(1+\exp(-0.8s))s+1} \tag{36}$$

which is obviously strongly unstable, yet formally stable, since $T(s)$ and $T_2(s)$ have the same spectrum (except for poles multiplicity). Hence, this algebraic operation (multiplication) preserves formal yet not strong stability. Recall, however, that formal stability will be tested by verification of strong stability, so there is some kind of conservativeness. ■

The crucial part of this section, the R_{MS} ring proposal, as a revisited and extended definition to the original one, follows.

Definition 2 (R_{MS} ring – a revision). An element $T(s)$ of R_{MS} ring is represented by a ratio of two (quasi)polynomials $y(s)/x(s)$ where the denominator is a (quasi)polynomial of degree n and the numerator can be factorized as

$$y(s) = \tilde{y}(s)\exp(-\tau s) \tag{37}$$

where $\tilde{y}(s)$ is a (quasi)polynomial of degree l and $\tau \geq 0$. Note that the degree of a quasipolynomial means its highest s-power.

The element lies in the space $H_\infty(\mathbb{C}^+)$, i.e. it is analytic and bounded in $\mathbb{C}^+$, particularly, there is no pole s_0 such that $\operatorname{Re} s_0 \geq 0$ for a retarded denominator or $\operatorname{Re} s_0 \geq -\varepsilon, \varepsilon > 0$ for a neutral one. If the term includes distributed delays, all roots of $x(s)$ in $\mathbb{C}^+$ are those of $y(s)$ (i.e. removable singularities). Moreover, $T(s)$ is formally stable. The strong stability condition (25) for (quasi)polynomial $x(s)$ is a sufficient but not necessary condition guaranteeing that.

In addition, the ratio is proper, i.e. $l \leq n$. More precisely, there exists a real number $R > 0$ for which holds that

$$\sup_{\operatorname{Re} s > 0, |s| \geq R} |T(s)| < \infty \tag{38}$$

see [28]. ■

5 Basic Properties of the Ring

5.1 Coprime factorization and Bézout identity

A basic operation on the quasipolynomial transfer function of TDS is coprime factorization by which the transfer function is decomposed into a coprime (or relatively prime) pair of ring elements. Since, in controller design, the intention is to use coprime factors in the Bézout equation (30), the factorization should also be Bézout, i.e. there must exists a stabilizing solution of (30) satisfying (29).

When dealing with coprime factorization, the divisibility condition has to be stated.

Lemma 1. (Divisibility in R_{MS}). Any $A(s) \in R_{MS}$ divides $B(s) \in R_{MS}$ if and only if all unstable zeros (including $s \to \infty$) of $A(s)$ are those of $B(s)$, and moreover, the numerator of $A(s)$ is formally stable. ■

Notice that zeros mean the roots of the whole term of the ring, not only those of the numerator.

Again, problems appear when dealing with neutral TDS or with those including distributed delays. An example of coprime, yet not Bézout, factorization of formally unstable neutral TDS was demonstrated in Example 3 and Example 5.

The following two examples demonstrate a typical coprime factorization over R_{MS} and a specific problem with distributed delays, respectively.

Example 7. The system is governed by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{s + \exp(-s)}{s^2 + (2 + \exp(-s))s + 1} \exp(-2s) \tag{39}$$

which is a stable retarded TDS. Coprime factorization of (39) over R_{MS} can be performed e.g. as follows

$$G(s) = \frac{\frac{b(s)}{m(s)}}{\frac{a(s)}{m(s)}} = \frac{\frac{(s + \exp(-s))\exp(-2s)}{m(s)}}{\frac{s^2 + (2 + \exp(-s))s + 1}{m(s)}} = \frac{B(s)}{A(s)} \tag{40}$$

where $A(s)$, $B(s) \in R_{MS}$ and $m(s)$ stands for a stable (quasi)polynomial of degree 2. Its degree must equal 2; otherwise, elements would not be proper or coprime. ■

Example 8. Consider a simple system with distributed delays with transfer function (16) and suggest a factorization

$$G(s) = \frac{1 - \exp(1)\exp(-s)}{s - 1} = \frac{\frac{1 - \exp(1)\exp(-s)}{m(s)}}{\frac{s-1}{m(s)}} = \frac{B(s)}{A(s)} \tag{41}$$

In this case, the common denominator (quasi)polynomial $m(s)$ could not be stable since it would lead to prime elements in R_{MS}. Indeed, let, for instance, $m(s) = s + 1$, then there exists a term $T(s) \in R_{MS}$ that is a non-zero non-invertible common divisor of both $A(s), B(s)$ (which are then reducible), e.g.

$$\begin{aligned} A(s) &= T(s)A_0(s) = \frac{s-1}{s+2}\frac{s+2}{s+1} \\ B(s) &= T(s)B_0(s) = \frac{s-1}{s+2}\frac{1 - \exp(1)\exp(-s)}{s-1} \end{aligned} \tag{42}$$

The solution of this problem is read as follows: The common denominator $m(s)$ must include all common zeros s_0 of $a(s), b(s)$ with $\operatorname{Re} s_0 \geq 0$ (even asymptotic ones tending to the imaginary axis). Thus, the coprime factorization (41) should read

$$G(s)=\frac{1-\exp(1)\exp(-s)}{s-1}=\frac{\dfrac{1-\exp(1)\exp(-s)}{s-1}}{\dfrac{s-1}{s-1}}=\frac{B(s)}{A(s)} \tag{43}$$

■

The notion of coprime factorization is closely related to the existence of a solution of the Bézout identity. As stated e.g. in Example 3, for formally unstable TDS such solution in $H_\infty(\mathbb{C}^+)$ (an thus not in R_{MS}) does not exist – we can obtain coprime yet not Bézout coprime factors.

If a pair $A(s), B(s) \in R_{MS}$ is Bézout coprime, it is possible to solve the Bézout identity (or to find the GCD) using the extended Euclidean algorithm. Prior to the implementation of the extended Euclidean algorithm to R_{MS} ring, an ordering of ring elements has to be defined, so that a poset is obtained. Thus, define $P=(R_{MS},\preceq)$ as

a) $A(s)\preceq B(s)$ iff $A(s)\,|\,B(s)$.

b) $A(s)\equiv B(s)$ iff $A(s)\,|\,B(s)$ and $B(s)\,|\,A(s)$, or equivalently, $A(s)$ is associated with $B(s)$.

c) $A(s)$ is not related to $B(s)$ iff $A(s)\nmid B(s)$ and $B(s)\nmid A(s)$.

The procedure of finding the $\text{GCD}(A(s),B(s))$ can be characterized as follows. Assume these three situations:

a) If $A(s)\equiv B(s)$, the GCD of both is simply either $A(s)$ or $B(s)$.

b) If $A(s)\succeq B(s)$, keep the following scheme

$$\left[\begin{array}{cc|c}1 & 0 & A(s)\\ 0 & 1 & B(s)\end{array}\right]\sim\left[\begin{array}{cc|c}1 & -\dfrac{A(s)}{B(s)} & 0\\ 0 & 1 & B(s)\end{array}\right] \tag{44}$$

hence, $B(s)$ is the GCD of $A(s)$ and $B(s)$, according to (2.46). If $B(s)\succeq A(s)$, the procedure is analogous with $\text{GCD}(A(s),B(s))= A(s)$.

c) Let $A(s)$ and $B(s)$ be not related to each other. In this case, follow the scheme (45).

Here, the GCD of $A(s)$ and $B(s)$ equals $A(s)X(s)+B(s)Y(s)$. In scheme (45), it is supposed that there can be found quotients $X(s),Y(s)$ such that the element $A(s)X(s)+B(s)Y(s)$ divides $A(s),B(s)$. Since $A(s),B(s)$ are Bézout coprime, $A(s)X(s)+B(s)Y(s)$ must be a unit of the ring.

$$\begin{aligned}&\left[\begin{array}{cc|c}1 & 0 & A(s)\\ 0 & 1 & B(s)\end{array}\right]\sim\left[\begin{array}{cc|c}X(s) & 0 & A(s)X(s)\\ 0 & 1 & B(s)\end{array}\right]\\ &\sim\left[\begin{array}{cc|c}X(s) & Y(s) & A(s)X(s)+B(s)Y(s)\\ 0 & 1 & B(s)\end{array}\right]\\ &\sim\left[\begin{array}{cc|c}0 & 1 & B(s)\\ X(s) & Y(s) & A(s)X(s)+B(s)Y(s)\end{array}\right]\\ &\sim\left[\begin{array}{cc|c}\dfrac{-B(s)X(s)}{A(s)X(s)+B(s)Y(s)} & \dfrac{A(s)X(s)}{A(s)X(s)+B(s)Y(s)} & 0\\ X(s) & Y(s) & A(s)X(s)+B(s)Y(s)\end{array}\right]\end{aligned} \tag{45}$$

In other words, the objective is to find structures of $X(s),Y(s)$ and to set zeros and poles of $A(s)X(s)+B(s)Y(s)$ such that divisibility conditions as in Lemma 1 are satisfied or the element is invertible. This task can be troublesome; however, if formally unstable neutral TDS were avoided being included, every numerator/denominator quasipolynomial would have only a finite number of unstable zeros, which would make possible to find the $\text{GCD}(A(s),B(s))$.

If the task is to solve the Bézout identity (30) itself instead of the $\text{GCD}(A(s),B(s))$, one can use scheme (9) where $c=1$. This yields these results, respectively

a) $$P(s)=\frac{1}{A(s)},\ Q(s)=0 \text{ and/or } P(s)=0,\ Q(s)=\frac{1}{B(s)} \tag{46}$$

b) $$P(s)=\frac{1}{A(s)},\ Q(s)=0 \text{ or } P(s)=0,\ Q(s)=\frac{1}{B(s)} \tag{47}$$

c) $$P(s)=\frac{X(s)}{A(s)X(s)+B(s)Y(s)},\quad Q(s)=\frac{Y(s)}{A(s)X(s)+B(s)Y(s)} \tag{48}$$

The following examples elucidate the whole procedure.

Example 9. Assume coprime factorization (43) and find $\text{GCD}(A(s),B(s))$ first. Since $A(s)$ divides $B(s)$, it holds that $B(s)\succeq A(s)$, hence

$$\mathrm{GCD}\left(A(s),B(s)\right)=A(s)=\frac{s-1}{s-1}=1 \quad (49)$$

according to (44).

The Bézout identity (30) then has the solution given by (47) as

$$P(s)=\frac{1}{A(s)}=1,\ Q(s)=0 \quad (50)$$

Example 10. Now let the factorization be given by (40) with $m(s)=(s+1)^2$. In this case, the both elements $A(s)$ and $B(s)$ are associated, thus $A(s)\equiv B(s)$ and scheme (45) can be used when solving $\mathrm{GCD}\left(A(s),B(s)\right)$. This scheme yields e.g.

$$\begin{aligned}&X(s)=Y(s)=1\\ &\Rightarrow A(s)X(s)+B(s)Y(s)\\ &=\frac{s^2+(2+\exp(-s))s+1+(s+\exp(-s))\exp(-2s)}{(s+1)^2}\\ &=\frac{s^2+(2+\exp(-s)+\exp(-2s)+\exp(-3s))s+1}{(s+1)^2}\\ &=\mathrm{GCD}(A(s),B(s))\end{aligned} \quad (51)$$

where $X(s)$, $Y(s)$ are chosen for the simplicity.

Then the solution of the Bézout identity according to (48) reads

$$\begin{aligned}&P(s)=Q(s)\\ &=\frac{(s+1)^2}{s^2+(2+\exp(-s)+\exp(-2s)+\exp(-3s))s+1}\end{aligned} \quad (52)$$

In case of asymptotically stable systems, i.e. $A(s)$ is invertible (a unit), it is possible to use also a simple procedure when solving the Bézout identity

$$Q(s)=1\Rightarrow P(s)=\frac{1-B(s)}{A(s)} \quad (53)$$

By applying this rule to the example, the following solution is obtained

$$P(s)=\frac{(s+1)^2-(s+\exp(-s))\exp(-2s)}{s^2+(2+\exp(-s))s+1} \quad (54)$$

This scheme has some advantages in controller design (this topic is out of the aim of this paper). ■

5.2 Ring properties

Follow now terms introduced in Section 2 and try to match some of them with R_{MS} ring.

Lemma 2. A set R_{MS} introduced in Definition 2 constitutes a commutative ring. ■

Proof. A sketch of proof that R_{MS} meets ring conditions follows.

Clearly, R_{MS} is closed under addition with associativity and the neutral element $E=0$. The inverse element $B(s)\in R_{MS}$ under addition of $A(s)\in R_{MS}$ is simply $B(s)=-A(s)$. Since $A(s)+B(s)=B(s)+A(s)\in R_{MS}$, it is a commutative group.

The closure under multiplication with associativity is also evident since the numerator and denominator of any $A(s)\in R_{MS}$ are composed of quasipolynomial factors – retarded ones and formally stable neutral ones, respectively. Since the operation of multiplication is commutative, left and right distributivity hold as well. In case of distributed delays, it is not possible to obtain more unstable denominator zeros then numerator ones of any $A(s)\in R_{MS}$ under multiplication. The multiplicative identity element equals 1. □

Lemma 11. An element $A(s)\in R_{MS}$ is a unit (invertible element) iff $A(s)$ has zero relative order and has the (asymptotically and formally) stable numerator. ■

The proof of Lemma 11 is evident (e.g. the necessity can be proved by the negation of the right hand side of the lemma) with the aid of Lemma 1. Note that stable numerator means that is has only stable zeros in the appropriate meaning.

Lemma 12. An element $A(s)\in R_{MS}$ is irreducible iff its numerator is formally stable and

$$O_R+N_U\le 1 \quad (55)$$

where O_R is the relative order and N_U stands for the number of real zeros $s_{U,i}, i=1,2,...N_U$ or conjugate pairs $s_{U,i},\bar{s}_{U,i}, i=1,2,...N_U$ with $\mathrm{Re}\, s_{U,i}\ge 0$ and $\mathrm{Re}\,\bar{s}_{U,i}\ge 0$ of $A(s)$, respectively. ■

Proof. Necessity. Consider the following three cases

a) $O_R=0$, $N_U=1$

b) $O_R=1$, $N_U=0$

c) $O_R \geq 2$

Use an indirect proof. First, let a) is not valid; hence, $O_R = 0$, $N_U > 1$. Consider a (quasi)polynomial $c(s)$ with only one unstable zero (or a pair of unstable zeros), say $c(s_{U,1}) = 0$ (or $c(s_{U,1}) = c(\bar{s}_{U,1}) = 0$) and an arbitrary stable (quasi)polynomial $b(s)$ of the same order (i.e. first or second one). Then

$$A(s) = \frac{a_{num}(s)}{a_{den}(s)} = \frac{a_{num}(s)b(s)}{a_{den}(s)c(s)} \frac{c(s)}{b(s)} = A_1(s)A_2(s) \quad (56)$$

where $A_1(s)$ and $A_2(s)$ are neither associated with $A(s)$ nor units.

Now, let b) is not valid, i.e. $O_R = 1$, $N_U > 0$, and assume a stable (quasi)polynomial $d(s)$ of the first order. Then follow the scheme

$$A(s) = \frac{a_{num}(s)}{a_{den}(s)} = \frac{a_{num}(s)d(s)}{a_{den}(s)} \frac{1}{d(s)} = A_1(s)A_2(s) \quad (57)$$

Again, $A_1(s)$ and $A_2(s)$ are neither associated with $A(s)$ nor units.

Finally, let c) holds. Then it is possible to write e.g. scheme (57).

Sufficiency. Consider the three cases introduced above again.

If a) holds and the numerator is formally stable (even asymptotically), scheme (56) fails, since $A_1(s)$ is a unit and $A_2(s)$ is associated with $A(s)$. Moreover, there is not possible to find another "reducible" scheme.

Similarly, if b) holds and is formally stable, $A_1(s)$ is a unit and $A_2(s)$ is associated with $A(s)$ in scheme (57); hence, $A(s)$ is irreducible. □

Lemma 13. R_{MS} ring does not constitute UFR. ■

Proof. Consider the following element of the ring

$$\frac{1-\exp(-\tau s)}{s} \quad (58)$$

Nonzero zeros of the numerator of (58) are

$$s_k = \frac{2k\pi}{\tau}\mathrm{j}, \bar{s}_k = -\frac{2k\pi}{\tau}\mathrm{j},\ k \in \mathrm{N} \quad (59)$$

Define polynomials

$$P_k(s) = (s - s_k)(s - \bar{s}_k) \quad (60)$$

Then the factorization

$$\begin{aligned} \frac{1-\exp(-\tau s)}{s} &= \frac{[1-\exp(-\tau s)](s+m_0)^2}{sP_1(s)} \frac{P_1(s)}{(s+m_0)^2} = \\ &= \frac{[1-\exp(-\tau s)](s+m_0)^4}{sP_1(s)P_2(s)} \frac{P_1(s)P_2(s)}{(s+m_0)^4} = \\ &\ldots \end{aligned} \quad (61)$$

where $m_0 > 0$ is infinite and thus the R_{MS} ring is not a UFR, and none of left-hand factors in (61) is irreducible and none of all factors is a unit. □

Lemma 14. R_{MS} is an integral domain. ■

Proof. Consider $A(s), B(s) \in R_{MS}$ where $A(s)$ is a unit. Let $A(s)B(s) = 0$ and multiply the whole equation by $1/A(s)$. It yields $B(s) = 0$ and we have a contradiction. □

Hence, Lemma 13 and Lemma 14 imply that R_{MS} is UFD.

Lemma 15. R_{MS} does not constitute PID. ■

Proof. Simply, it holds that every PID is UFD. Since R_{MS} is not UFD according to Lemma 13, it is not PID. □

Lemma 16. R_{MS} does not constitute a Bézout domain. ■

Proof. It is sufficient to show that there exists a pair $A(s), B(s) \in R_{MS}$ which does not give a solution pair $Q(s), P(s) \in R_{MS}$ of (30). Indeed, as mentioned above, coprime factorization of formally unstable TDS does not have a stabilizing solution of the Bézout identity in $H_\infty(\mathrm{C}^+)$, i.e. condition (29) does not hold. Since $H_\infty(\mathrm{C}^+) \supset R_{MS}$, which is evident from Definition 2, such solution does not exist in R_{MS} as well. □

The decision whether R_{MS} is a Noetherian ring is not successfully solved. Typically, a ring is a Bézout domain yet not PID, i.e. there exists an infinitely generated ideal which is not principal. In such cases, the ring is not Noetherian, see e.g. ring $\mathcal{E}$ of pseudopolynomials or ring $\mathcal{H}$, see Section 3.

6 Conclusions

The presented paper has introduced the original and a revised (alternative) definition of a special algebraic structure (ring) of quasipolynomial meromorphic functions. After offering an acquaintance with basic algebraic notions, an overview of some algebraic analytic and control structures and methods has been given. The original definition of R_{MS} has followed and some its disadvantages have been mentioned. Thus, a proposition of a revised definition has been then introduced, which is the crucial part of this contribution. The most involved part of the paper, i.e. (Bézout) coprime factorization, issues about the solution of the Bézout identity in the ring and selected algebraic properties, has followed.

As mentioned above several times, the ring can be used not only for TDS description but primarily for algebraic controller design satisfying asymptotic and formal stability of a control feedback system, reference tracking, asymptotic load disturbance rejection, etc., see e.g. [36], [37]. To comprehend this broad topic, some preliminary and supporting problems had to be analyzed and solved, for instance [38]-[42].

The natural limitation of the methodology is that formally unstable neutral TDS can not be stabilized in the sense of the ring. A detailed description of this control approach in the revised ring will be the matter of a future paper.

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